DELTA SHOCK WAVE INTERACTIONS VIA WAVE FRONT TRACKING METHOD

NEBOJŠA DEDOVIĆ AND MARKO NEDELJKOV

Abstract. In this paper we discuss delta shock interaction problem for a pressureless gas dynamics system with two different ways of approaching the subject. The first one is by using shadow wave solution concept. The result of two delta shock interactions is delta shock with non-constant speed in a general case. The second one is by perturbing the system with a small pressure term. The obtained perturbed system is strictly hyperbolic and its Riemann problem is solvable. We compare a limit of a numerical wave front tracking results as small pressure term vanishes with the shadow wave solution.

Key words: weighted shadow waves, delta shock waves, wave front tracking, Riemann problem, interactions

1. Introduction

Consider the one-dimensional Euler gas dynamics system given by

$$\begin{aligned} \partial_t\rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p(\varepsilon,\rho)) &= 0, \end{aligned} \tag{1}$$

where ρ is the density, $m = \rho\, u$ is the momentum, $p(\varepsilon,\rho) = \varepsilon\, p_0(\rho)$ is the scalar pressure, $\varepsilon << 1$ and $p_0(\rho) = \rho^\gamma/\gamma$. Taking $\varepsilon \to 0$ in (1), we obtain the pressureless gas dynamics model (PGD model in the rest of the paper), also called sticky particles model (in [13])

$$\begin{aligned} \partial_t\rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) &= 0,\ (x,t) \in \mathbb{R}\times\mathbb{R}_+\,. \end{aligned} \tag{2}$$

System (1) can be considered as a perturbation of system (2) which is weakly hyperbolic with a double eigenvalue $\lambda_1 = \lambda_2 = u$. All entropy pairs (η, q) with a semiconvex function η are given by $\eta := \rho S(u)$, $q := \rho\, u S(u)$, where $S'' \geq 0$ (the entropy function η is semi-convex with respect to the variable $(\rho, \rho u)$). The Riemann problem

$$\rho(x,0) = \begin{cases} \rho_0, & x<0\,, \\ \rho_1, & x>0\,, \end{cases}\,, \qquad u(x,0) = \begin{cases} u_0, & x<0\,, \\ u_1, & x>0\,, \end{cases} \tag{3}$$

has a classical entropy solution consisting of two contact discontinuities connected with the vacuum state $(\rho = 0)$ if $u_0 \leq u_1$:

$$(\rho(x,t), u(x,t)) = \begin{cases} (\rho_0, u_0), & x < u_0 t, \\ (0, \psi(x/t)), & u_0 t < x < u_1 t, \\ (\rho_1, u_1), & x > u_1 t, \end{cases}$$

where $\psi(y) = y$. We are now turning to the case $u_0 > u_1$ when there is no classical solution to the Riemann problem (2, 3).

Throughout this paper, the following constants will be fixed:

$$\gamma = 1 + 2\varepsilon,\ 0 < \varepsilon < \frac{1}{2},\ \kappa = \frac{\sqrt{\varepsilon}}{\sqrt{\gamma}} \text{ and } p = \kappa^2 \rho^\gamma \ . \tag{4}$$

2. Elementary waves of the perturbed system

The eigenvalues of system (1) are

$$\begin{aligned} \lambda_1 &= u - \kappa\sqrt{\gamma}\rho^{\frac{\gamma-1}{2}}, \\ \lambda_2 &= u + \kappa\sqrt{\gamma}\rho^{\frac{\gamma-1}{2}}, \end{aligned} \tag{5}$$

and the corresponding eigenvectors are

$$\begin{aligned} r_1 &= (-1, -u + \kappa\sqrt{\gamma}\rho^{\frac{\gamma-1}{2}})^T, \\ r_2 &= (1, u + \kappa\sqrt{\gamma}\rho^{\frac{\gamma-1}{2}})^T \ . \end{aligned} \tag{6}$$

We have chosen an orientation such that $\nabla\lambda_i \cdot r_i > 0,\ i = 1, 2$, since both fields are genuinely nonlinear. The corresponding Riemann invariants of system (1) are

$$\begin{array}{r} s = u + \frac{\kappa\sqrt{\gamma}}{\varepsilon}(\rho^\varepsilon - 1): \text{ 1-invariant, and} \\ r = u - \frac{\kappa\sqrt{\gamma}}{\varepsilon}(\rho^\varepsilon - 1): \text{ 2-invariant}. \end{array} \tag{7}$$

The rarefaction curves through the point (ρ_0, u_0) are given by

$$\begin{array}{r} u - u_0 = -\frac{\kappa\sqrt{\gamma}}{\varepsilon}(\rho^\varepsilon - \rho_0^\varepsilon)\ ,\ \ 0 \le \rho \le \rho_0: \text{ 1-rarefaction curve,} \\ u - u_0 = \ \ \frac{\kappa\sqrt{\gamma}}{\varepsilon}(\rho^\varepsilon - \rho_0^\varepsilon)\ ,\ \ \rho \ge \rho_0: \text{ 2-rarefaction curve,} \end{array} \tag{8}$$

while the shock curves through the point (ρ_0, u_0) are given by

$$u - u_0 = -\kappa\sqrt{\frac{\rho^\gamma - \rho_0^\gamma}{\rho_0\rho(\rho - \rho_0)}}\,(\rho - \rho_0), \quad \rho > \rho_0: \text{ 1-shock curve,} \tag{9}$$

and

$$u - u_0 = \kappa\sqrt{\frac{\rho^\gamma - \rho_0^\gamma}{\rho_0\rho(\rho - \rho_0)}}\,(\rho - \rho_0), \quad 0 < \rho < \rho_0: \text{ 2-shock curve.} \tag{10}$$

With the Riemann invariants, shock curves starting from the point (r_0, s_0) are

$$S_1: \begin{cases} r_0 - r = \kappa\rho_0^\varepsilon\left(\sqrt{\dfrac{(\alpha-1)(\alpha^\gamma-1)}{\alpha}} + \sqrt{\gamma}\dfrac{\alpha^\varepsilon - 1}{\varepsilon}\right), \\ s_0 - s = \kappa\rho_0^\varepsilon\left(\sqrt{\dfrac{(\alpha-1)(\alpha^\gamma-1)}{\alpha}} - \sqrt{\gamma}\dfrac{\alpha^\varepsilon - 1}{\varepsilon}\right), \end{cases} \tag{11}$$

where $r_0 = r(\rho_0, u_0)$, $s_0 = s(\rho_0, u_0)$ and $\alpha = \rho/\rho_0 \ge 1$, and

$$S_2: \begin{cases} s_0 - s = \kappa\rho_0^\varepsilon\left(\sqrt{\dfrac{(1-\alpha)(1-\alpha^\gamma)}{\alpha}} + \sqrt{\gamma}\dfrac{1 - \alpha^\varepsilon}{\varepsilon}\right), \\ r_0 - r = \kappa\rho_0^\varepsilon\left(\sqrt{\dfrac{(1-\alpha)(1-\alpha^\gamma)}{\alpha}} - \sqrt{\gamma}\dfrac{1 - \alpha^\varepsilon}{\varepsilon}\right), \end{cases} \tag{12}$$

where $r_0 = r(\rho_0, u_0)$, $s_0 = s(\rho_0, u_0)$ and $0 < \alpha = \rho/\rho_0 \leq 1$. The corresponding rarefaction curves are given by

$$R_1 : \quad r \geq r_0,\ s = s_0, \tag{13}$$

and

$$R_2 : \quad s \geq s_0,\ r = r_0. \tag{14}$$

It is clear that from (11, 12) we have that $r_0 - r \geq s_0 - s$ holds for S_1 curve and $s_0 - s \geq r_0 - r$ holds for S_2 curve, respectively.

The Riemann problem for system (1) with initial data (3) was solved by Riemann [12], and the result is summarized in the following theorem (the proof can be found in Courant-Friedrichs [4] and Smoller [14]).

Theorem 2.1. [1] *Consider system (1) with initial data (3). Suppose that* $u_1 - u_0 < \frac{\kappa\sqrt{\gamma}}{\varepsilon}(\rho_1^\varepsilon + \rho_0^\varepsilon)$, *or equivalently* $s_0 - r_1 > -\frac{2\kappa\sqrt{\gamma}}{\varepsilon}$. *Then there exists a unique solution composed of constant states* $(\rho_0, u_0) = (r_0, s_0)$, $(\rho_m, u_m) = (r_m, s_m)$ *and* $(\rho_1, u_1) = (r_1, s_1)$ *separated by centered rarefaction or shock waves satisfying the following estimates:*

$$\begin{aligned} r(x,t) = r(\rho(x,t), u(x,t)) \geq \min\{r_0, r_1\}, \\ s(x,t) = s(\rho(x,t), u(x,t)) \leq \max\{s_0, s_1\}. \end{aligned} \tag{15}$$

The amplitude of the waves is denoted by

$$\begin{aligned} \beta := r_m - r_0 \quad &: \ \textit{amplitude of an 1-wave}, \\ \chi := s_1 - s_m \quad &: \ \textit{amplitude of a 2-wave}. \end{aligned} \tag{16}$$

Here $\beta, \chi \geq 0$ *for centered rarefaction waves and* $\beta, \chi < 0$ *for shock waves; absolute values* $|\beta|$, $|\chi|$ *are called strengths of* β *and* χ, *respectively.*

We shall use that notation throughout the rest of the paper.

3. Local Interactions Estimates

Our first task is to obtain a sharp estimate of wave strengths with respect to ε as much as possible. In order to do that, we shall present some assertions from [11] together with modified proofs, since certain changes in estimates will be useful for our investigation.

Theorem 3.1. [11] *The shock curve* S_1 *starting at the point* (r_0, s_0) *is given by*

$$s_0 - s = g_1(r_0 - r, \rho_0) = \int_0^{r_0 - r} h_1(\alpha)|_{\alpha = \alpha_1(\beta/\kappa\rho_0^\varepsilon)}\, d\beta, \quad r < r_0, \tag{17}$$

where $0 \leq g_1'(\beta, \rho_0) < 1$ *and* $g_1''(\beta, \rho_0) \geq 0$[1]. *The shock curve* S_2 *starting at the point* (r_0, s_0) *is*

$$r_0 - r = g_2(s_0 - s, \rho_0) = \int_0^{s_0 - s} h_2(\alpha)|_{\alpha = \alpha_2(\chi/\kappa\rho_0^\varepsilon)}\, d\chi, \quad s < s_0, \tag{18}$$

where $0 \leq g_2'(\chi, \rho_0) < 1$ *and* $g_2''(\chi, \rho_0) \geq 0$.

[1]The primes denote differentiation with respect to the first argument.

Proof. We shall repeat the proof from [11] in order to fix the notation for the rest of the paper. Relation $s_0 - s = g_1(r_0 - r, \rho_0)$ implies

$$\frac{\partial(s_0 - s)}{\partial\alpha} = \frac{\partial g_1(r_0 - r, \rho_0)}{\partial(r_0 - r)} \cdot \frac{\partial(r_0 - r)}{\partial\alpha}, \text{ so } \frac{\partial(s_0 - s)/\partial\alpha}{\partial(r_0 - r)/\partial\alpha} = g_1'(\beta, \rho_0)\,. \tag{19}$$

If

$$h_1(\alpha) = \frac{\partial(s_0 - s)/\partial\alpha}{\partial(r_0 - r)/\partial\alpha}\,,$$

then one can easily see that

$$h_1(\alpha) = \left(\frac{Y-1}{Y+1}\right)^2 \quad \text{with} \quad Y = \sqrt{\frac{\gamma\alpha^\gamma(\alpha - 1)}{\alpha^\gamma - 1}}\,, \quad \text{for} \quad \alpha > 1\,. \tag{20}$$

From the first equation in (11), we have

$$\frac{\beta}{\kappa\rho_0^\varepsilon} = \sqrt{\frac{(\alpha-1)(\alpha^\gamma - 1)}{\alpha}} + \sqrt{\gamma}\,\frac{\alpha^\varepsilon - 1}{\varepsilon} =: f(\alpha)\,. \tag{21}$$

Therefore

$$f'(\alpha) > \quad \frac{1}{2}\sqrt{\frac{\alpha}{(\alpha-1)(\alpha^\gamma - 1)}} \cdot \frac{\alpha^\gamma - 1}{\alpha^2} + \sqrt{\gamma}\,\alpha^{\varepsilon - 1} > 0$$

since $\alpha^\gamma > 1$ for $\alpha > 1$ and $\gamma > 1$. Using the fact that $f'(\alpha) > 0$ and (21) the Implicit Function Theorem yields that there exists $\alpha = \alpha_1(\beta/\kappa\rho_0^\varepsilon)$ such that

$$g_1(r_0 - r, \rho_0) = \int_0^{r_0 - r} h_1(\alpha)|_{\alpha = \alpha_1(\beta/\kappa\rho_0^\varepsilon)}\, d\beta\,. \tag{22}$$

Since $g_1'(\beta, \rho_0) = h_1(\alpha)$, $g_1''(\beta, \rho_0) = h_1'(\alpha) \cdot \dfrac{d\alpha}{d\beta}$ and $\dfrac{d\beta}{d\alpha} = \kappa\rho_0^\varepsilon \cdot f'(\alpha) > 0$ it remains to prove that $0 \le h_1(\alpha) < 1$ and $0 \le h_1'(\alpha)$. From (20) we have

$$0 \le h_1(\alpha) = \left(\frac{Y-1}{Y+1}\right)^2 < \left(\frac{Y+1}{Y+1}\right)^2 = 1\,,$$

and

$$0 \le h_1'(\alpha) = 4 \cdot \frac{Y-1}{(Y+1)^3} \cdot Y'\,, \tag{23}$$

since $Y \ge 1$ and $Y' \ge 0$. The second part of the theorem can be proved using the same technique. $\square$

Lemma 3.2. *Let $\rho_0 < \rho_1$ and $\beta/\kappa\rho_1^\varepsilon < \theta < \beta/\kappa\rho_0^\varepsilon$. Then*

$$\frac{d\alpha}{d\theta} = \frac{1}{f'(\alpha)} = \frac{2Y}{\sqrt{\gamma}\alpha^{\frac{\gamma-3}{2}}(1+Y)^2}\,. \tag{24}$$

We would need an estimate of the difference of Riemann invariants across two shock waves which is more precise than the one in [11]. It is provided by the following theorem.

Theorem 3.3. *Let $0 < \varepsilon < \frac{1}{2}$, $s_0 < s_1$, and take two S_1 curves originating at the points $(r_0, s_0) = (\rho_0, u_0)$ and $(r_0, s_1) = (\rho_1, u_1)$, which are continued to the points (r, s) and (r, s_2), respectively. Then we have*

$$0 \le (s_0 - s) - (s_1 - s_2) \le C_*\sqrt{\varepsilon}\,(r_0 - r)\,(s_1 - s_0)\,, \tag{25}$$

where C_* is a constant independent of ε, ρ_0 and ρ_1.

Proof. Let $z^0 = s_0 - s$, $z^1 = s_1 - s_2$ and $w = r_0 - r$ (look at the diagram shown in Figure 1).

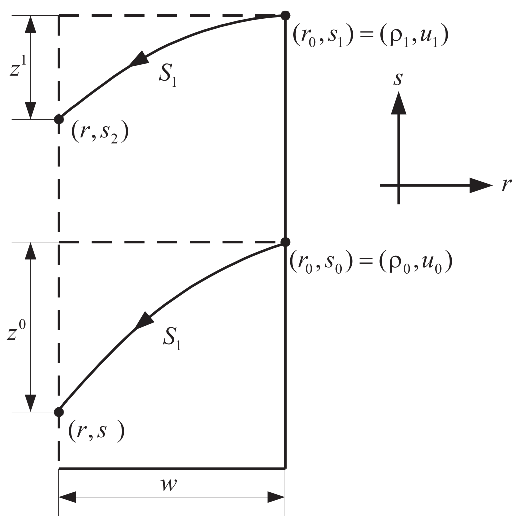

FIGURE 1. Two 1-shock wave curves in $r - s$ plane.

By Theorem 3.1 and the Mean Value Theorem we know that for $\rho_1 > \rho_0$, there exists θ such that

$$z^0 - z^1 = \int_0^w \frac{dh_1(\alpha)}{d\alpha}\Big|_{\alpha=\alpha(\theta)} \cdot \alpha'(\theta) \left(\frac{\beta}{\kappa\rho_0^\varepsilon} - \frac{\beta}{\kappa\rho_1^\varepsilon} \right) d\beta \, , \tag{26}$$

where $\theta \in \left(\frac{\beta}{\kappa\rho_1^\varepsilon}, \frac{\beta}{\kappa\rho_0^\varepsilon} \right)$. The definitions of h_1 and α imply

$$\frac{dh_1(\alpha)}{d\alpha} \geq 0, \quad \frac{d\alpha(\theta)}{d\theta} \geq 0 \quad \text{and} \quad \frac{\beta}{\kappa\rho_0^\varepsilon} - \frac{\beta}{\kappa\rho_1^\varepsilon} \geq 0$$

so $z^0 - z^1 \geq 0$ for $\rho_1 > \rho_0$. We need to estimate the integrand in (26). By (20), we have

$$\frac{dh_1(\alpha)}{d\alpha} \cdot \frac{d\alpha(\theta)}{d\theta} \leq \frac{4(Y-1)(\gamma+1)\alpha^{\frac{1-\gamma}{2}}}{\sqrt{\gamma}\,(Y+1)^3} \, .$$

Thus,

$$z^0 - z^1 \leq \frac{4(\gamma+1)}{\kappa\sqrt{\gamma}\,\rho_0^\varepsilon\rho_1^\varepsilon}(\rho_1^\varepsilon - \rho_0^\varepsilon) \int_0^w \beta\, \alpha^{\frac{1-\gamma}{2}}\Big|_{\alpha=\alpha(\theta)} \frac{Y-1}{(Y+1)^3}\, d\beta \tag{27}$$

and

$$z^0 - z^1 \leq \frac{4(\gamma+1)}{\kappa\sqrt{\gamma}\,\rho_0^\varepsilon\rho_1^\varepsilon}(\rho_1^\varepsilon - \rho_0^\varepsilon) \int_0^w \beta\, \alpha^{-\frac{1+\gamma}{2}}\Big|_{\alpha=\alpha(\theta)} d\beta \, . \tag{28}$$

From Lemma 3.2 we know that $d\alpha/d\theta > 0$ for $\beta/\kappa\rho_1^\varepsilon < \theta < \beta/\kappa\rho_0^\varepsilon$. Hence,

$$\alpha\left(\frac{\beta}{\kappa\rho_1^\varepsilon}\right) \leq \alpha\,(\theta) \leq \alpha\left(\frac{\beta}{\kappa\rho_0^\varepsilon}\right). \tag{29}$$

Moreover,

$$\frac{\beta}{\kappa\rho_1^\varepsilon} = f(\alpha) \leq \quad 2\sqrt{\frac{(\alpha-1)\,\alpha^\gamma}{\alpha-1}} = 2\alpha^{\gamma/2}\ , \text{ for } \alpha = \alpha\left(\frac{\beta}{\kappa\rho_1^\varepsilon}\right)\ .$$

At this point, we use a majorization of $z^0 - z^1$ different from the one in [11] in order to obtain bounds for C_* independent of ε. By (29) and the above inequality we obtain

$$\begin{aligned}
&\frac{\beta}{2\kappa\rho_1^\varepsilon} \leq \left(\alpha\left(\frac{\beta}{\kappa\rho_1^\varepsilon}\right)\right)^{\gamma/2} \Rightarrow \left(\frac{\beta}{2\kappa\rho_1^\varepsilon}\right)^{-\frac{\gamma+1}{\gamma}} \geq \left(\alpha\left(\frac{\beta}{\kappa\rho_1^\varepsilon}\right)\right)^{-\frac{\gamma+1}{2}}\ , \ \text{ so} \\
&\min\left\{1, \left(\frac{\beta}{2\kappa\rho_1^\varepsilon}\right)^{-\frac{\gamma+1}{\gamma}}\right\} \geq \left(\alpha\left(\frac{\beta}{\kappa\rho_1^\varepsilon}\right)\right)^{-\frac{\gamma+1}{2}} \geq (\alpha(\theta))^{-\frac{\gamma+1}{2}}\ ,
\end{aligned} \tag{30}$$

for $\alpha\left(\frac{\beta}{\kappa\rho_1^\varepsilon}\right) \geq 1$ and $\gamma > 1$. Since $d\alpha/d\theta > 0$, it follows by (28) that

$$\begin{aligned}
z^0 - z^1 \leq\ & \frac{4(\gamma+1)}{\kappa\sqrt{\gamma}\,\rho_0^\varepsilon\rho_1^\varepsilon}(\rho_1^\varepsilon - \rho_0^\varepsilon)\int_0^w \beta\cdot\min\left\{1, \left(\frac{\beta}{2\kappa\rho_1^\varepsilon}\right)^{-\frac{\gamma+1}{\gamma}}\right\}\,d\beta \\
\leq\ & \frac{8(\gamma+1)}{\sqrt{\gamma}\,\rho_0^\varepsilon}\,(\rho_1^\varepsilon - \rho_0^\varepsilon)\int_0^w \min\left\{\frac{\beta}{2\kappa\rho_1^\varepsilon}, \left(\frac{\beta}{2\kappa\rho_1^\varepsilon}\right)^{-\frac{1}{\gamma}}\right\}\,d\beta \\
\leq\ & \frac{8(\gamma+1)}{\sqrt{\gamma}\,\rho_0^\varepsilon}\,(\rho_1^\varepsilon - \rho_0^\varepsilon)\,w\,.
\end{aligned} \tag{31}$$

Using

$$\rho_1^\varepsilon - \rho_0^\varepsilon = \frac{\varepsilon}{2\kappa\sqrt{\gamma}}(s_1 - s_0)$$

and $\kappa\sqrt{\gamma} = \sqrt{\varepsilon}$ together with (31) we finally get

$$\begin{aligned}
z^0 - z^1 \leq\ & \frac{4(\gamma+1)}{\sqrt{\gamma}\,\rho_0^\varepsilon}\cdot\frac{\varepsilon}{\kappa\sqrt{\gamma}}\,(s_1 - s_0)\,w \\
=\ & \frac{4(\gamma+1)}{\sqrt{\gamma}\,\rho_0^\varepsilon}\cdot\sqrt{\varepsilon}\,(s_1 - s_0)\,w\,.
\end{aligned} \tag{32}$$

Suppose that ρ is the first component of the solution of the Riemann problem (1, 3) for $u_0 > u_1$. Then Lemma 3.1 from [3] yields that for small $\varepsilon > 0$, there exists $C > 0$ independent of ε, such that $\rho \leq C/\varepsilon$. Now, using (32), if $\rho_0 \sim 1/\varepsilon$ then $\rho_0^\varepsilon \sim 1$ as $\varepsilon \to 0$. For ε small enough we may write $\rho_0^\varepsilon \geq C_1$. Thus, there exists a constant C_* independent of κ, ρ_0, ρ_1 and ε such that

$$z^0 - z^1 \leq C_*\,\sqrt{\varepsilon}\,(s_1 - s_0)\,w, \tag{33}$$

holds. This completes the proof of the theorem. □

The theorem that follows can be proved in the same way.

Theorem 3.4. *Let $0 < \varepsilon < \frac{1}{2}$, $r_0 > r_1$, and take two S_2 curves originating at the points $(r_0, s_0) = (\rho_0, u_0)$ and $(r_1, s_0) = (\rho_1, u_1)$, which are continued to the points (r, s) and (r_2, s), respectively. Then we have*

$$0 \leq (r_0 - r) - (r_1 - r_2) \leq C_{**}\,\sqrt{\varepsilon}\,(s_0 - s)\,(r_0 - r_1)\,, \tag{34}$$

where C_{**} is a constant independent of ε, ρ_0 and ρ_1.

(A1) We shall use the following convention: C_* denotes the maximum of the constants C_* and C_{**} from Theorems 3.3 and 3.4, respectively.

In the following theorem β and χ denote S_1 and S_2, respectively, while o and π denote R_1 and R_2, respectively. The prime is reserved for after interaction waves. (For example, the interaction of S_2 and S_1 which produces S_1 and S_2 is denoted by $\chi + \beta \to \beta' + \chi'$.)

Theorem 3.5. *If $0 < \varepsilon < \frac{1}{2}$, then the following estimates are valid for the corresponding interactions:*

(1) *S_2 and S_1 interaction:*
  (a) $\chi + \beta \to \beta' + \chi'$
  $|\beta'| \le |\beta| + C_*\sqrt{\varepsilon}\,|\chi||\beta|, \quad |\chi'| \le |\chi| + C_*\sqrt{\varepsilon}\,|\beta||\chi|$, *or*

  there exist η, ξ such that
  (b) $\chi + \beta \to \beta' + \chi'$
  $0 \le |\beta'| = |\beta| - \xi, \quad |\chi'| \le |\chi| + C_*\sqrt{\varepsilon}\,|\beta||\chi| + \eta,$
  where $0 \le \eta \le g_1'(|\beta|, \rho_0)\xi < \xi$, or

  (c) $\chi + \beta \to \beta' + \chi'$
  $0 \le |\chi'| = |\chi| - \xi, \quad |\beta'| \le |\beta| + C_*\sqrt{\varepsilon}\,|\chi||\beta| + \eta,$
  where $0 \le \eta \le g_1'(|\chi|, \rho_0)\xi < \xi$.

(2) *S_2 and R_1 (or R_2 and S_1) interaction:*
  (a) $\chi + o \to o' + \chi'$
  $|\chi'| = |\chi|, \quad |o'| \le |o| + C_*\sqrt{\varepsilon}\,|\chi||o|$.
  (b) $\pi + \beta \to \beta' + \pi'$
  $|\beta'| = |\beta|, \quad |\pi'| \le |\pi| + C_*\sqrt{\varepsilon}\,|\beta||\pi|$.

(3) *S_2 and S_2 (or S_1 and S_1) interaction:*
  (a) $\chi_1 + \chi_2 \to o' + \chi'$:
  $|\chi'| = |\chi_1| + |\chi_2|, \quad |o'| \le |\chi_1| + |\chi_2|$.
  (b) $\beta_1 + \beta_2 \to \beta' + \pi'$:
  $|\beta'| = |\beta_1| + |\beta_2|, \quad |\pi'| \le |\beta_1| + |\beta_2|$.

(4) *S_2 and R_2 (or R_1 and S_1) interaction:*
  (a) 1° $\chi + \pi \to \beta' + \chi'$: *there exist 1-shock β_0 and 2-shock χ_0 such that*
  $|\chi_0| = |\chi| - \xi, \ |\beta_0| = \eta$ *and* $\chi_0 + \beta_0 \to \beta' + \chi'$,
  where $0 < \eta \le g_2'(|\chi|, \rho_1)\xi < \xi$.
  2° $\chi + \pi \to \beta' + \pi'$: *there exist η, ξ such that*
  $|\pi'| \le |\pi|, \quad |\beta'| = \eta < \xi = |\chi|,$
  where $0 < \eta \le g_2'(|\chi|, \rho_1)\xi < \xi$.
  (b) 1° $o + \beta \to \beta' + \chi'$: *there exist 1-shock β_0 and 2-shock χ_0 such that*
  $|\beta_0| = |\beta| - \xi, \ |\chi_0| = \eta$ *and* $\chi_0 + \beta_0 \to \beta' + \chi'$,
  where $0 < \eta \le g_2'(|\beta|, \rho_2)\xi < \xi$.
  2° $o + \beta \to o' + \chi'$
  $|o'| \le |o|, \quad |\chi'| = \eta < \xi = |\beta|,$
  where $0 < \eta \le g_1'(|\beta|, \rho_0)\xi < \xi$.

(5) *R_2 and S_2 (or S_1 and R_1) interaction:*
  (a) 1° $\pi + \chi \to \beta' + \chi'$: *there exist η, ξ such that*
  $|\chi'| = |\chi| - \xi, \quad |\beta'| = \eta,$

where $0 < \eta \le g_1'(|\chi|, \rho_2)\xi < \xi$.
2° $\pi + \chi \to \beta' + \pi'$
$|\pi'| \le |\pi|, \quad |\beta'| = \eta < \xi = |\chi|,$
where $0 < \eta \le g_2'(|\chi|, \rho_0)\xi < \xi$.
(b) 1° $\beta + o \to \beta' + \chi'$: *there exist* η, ξ *such that*
$|\beta'| = |\beta| - \xi, \quad |\chi'| = \eta,$
where $0 < \eta \le g_1'(|\beta|, \rho_1)\xi < \xi$.
2° $\beta + o \to o' + \chi'$
$|o'| \le |o|, \quad |\chi'| = \eta < \xi = |\beta|,$
where $0 < \eta \le g_1'(|\beta|, \rho_1)\xi < \xi$.
(6) R_2 *and* R_1 *interaction:*
$\pi + o \to o' + \pi'$
$|o'| = |o|, \quad |\pi'| = |\pi|.$

Here C_* *is a positive constant defined as in (A1).*

Proof. This theorem can be proved using the same tools as in [11] and therefore will be omitted. The only differences are: the constant C_* is now independent of $\varepsilon, \beta, \chi, \rho_0, \rho_1$ and ρ_2, and we have $\sqrt{\varepsilon}$ instead of ε in the estimates $(1)(a) - (1)(c)$, $(2)(a)$ and $(2)(b)$. □

The main part of the paper is the interaction problem of delta shocks via pressure perturbation. Thus, one needs to control shock and rarefaction strengths as ρ goes to infinity as $\varepsilon \to 0$ (more precisely, when ρ is bounded by const/ε). Because of that, we give their estimates in $r - s$ plane based on Theorem 3.3 and Theorem 3.4. Let $(\rho_0, u_0) = (r_0, s_0)$ be connected with $(\rho, u) = (r, s)$ by a 1-rarefaction (or 1-shock) wave, while $(\rho, u) = (r, s)$ be connected with $(\rho_1, u_1) = (r_1, s_1)$ by a 2-rarefaction (or 2-shock) wave. Then the strength of 1-rarefaction wave is

$$r - r_0 = \frac{2}{\sqrt{\varepsilon}}(\rho_0^\varepsilon - \rho^\varepsilon), \quad \rho < \rho_0\,, \tag{35}$$

and the strength of 2-rarefaction wave is

$$s_1 - s = \frac{2}{\sqrt{\varepsilon}}(\rho_1^\varepsilon - \rho^\varepsilon), \quad \rho < \rho_1\,. \tag{36}$$

The strength of 1-shock wave is estimated by

$$2\,\rho_0^\varepsilon\,\sqrt{\varepsilon}\,\ln\frac{\rho}{\rho_0} \le r_0 - r \le 2\,\frac{\sqrt{\varepsilon}}{\sqrt{1+2\varepsilon}}\,\left(\frac{\rho}{\rho_0}\right)^{\gamma/2}\cdot\rho_0^\varepsilon, \quad \rho > \rho_0\,, \tag{37}$$

while, the strength of 2-shock wave is estimated by

$$2\,\rho_1^\varepsilon\,\sqrt{\varepsilon}\,\ln\frac{\rho}{\rho_1} \le s - s_1 \le 2\,\frac{\sqrt{\varepsilon}}{\sqrt{1+2\varepsilon}}\,\left(\frac{\rho}{\rho_1}\right)^{\gamma/2}\cdot\rho_1^\varepsilon, \quad \rho > \rho_1\,. \tag{38}$$

Let us estimate the upper bound of the 1-shock wave given in (37). For the function g_1 from (17) we have $0 \le g_1'(\beta, \rho_0) < 1$ and $0 \le g_1''(\beta, \rho_0)$, so

$$\lim_{|\beta| \to +\infty} g_1'(|\beta|, \rho_0) \le 1.$$

Let us consider two special cases needed for our investigation. The first case: $\rho > \rho_0$ and $\rho \sim 1/\varepsilon$. We have that there exist constants $\tilde{C}$, $\bar{\bar{C}}$ and $\bar{C}$ independent of ε such

that

$$\left(\frac{\rho}{\rho_0}\right)^{\gamma/2} \cdot \rho_0^\varepsilon = \sqrt{\frac{\rho}{\rho_0}} \cdot \rho^\varepsilon \leq \sqrt{\frac{\tilde{C}}{\varepsilon}} \cdot \left(\frac{\bar{\bar{C}}}{\varepsilon}\right)^\varepsilon \leq \sqrt{\frac{1}{\varepsilon}} \cdot \bar{C}, \text{ so}$$

$$2\,\frac{\sqrt{\varepsilon}}{\sqrt{1+2\varepsilon}} \left(\frac{\rho}{\rho_0}\right)^{\gamma/2} \cdot \rho_0^\varepsilon \leq 2\,\frac{\sqrt{\varepsilon}}{\sqrt{1+2\varepsilon}} \cdot \frac{\bar{C}}{\sqrt{\varepsilon}} \leq \text{const.}$$

It follows that there exists a constant C_2, independent of ε and ρ_0, such that

$$\sup g_1'(|\beta|, \rho_0) := C_2 < 1\,. \tag{39}$$

Hence,

$$\frac{1 - g_1'(|\beta|, \rho_0)}{g_1'(|\beta|, \rho_0)} \geq \frac{1 - C_2}{C_2} =: C_3 > 0\,. \tag{40}$$

The second case: $\rho > \rho_0$, $\rho \sim 1/\varepsilon$ and $\rho_0 \sim 1/\varepsilon$. Then

$$\sqrt{\frac{\rho}{\rho_0}} \cdot \rho^\varepsilon \sim \text{const} \Rightarrow 2\,\frac{\sqrt{\varepsilon}}{\sqrt{1+2\varepsilon}} \sqrt{\frac{\rho}{\rho_0}}\, \rho^\varepsilon \sim \mathcal{O}(\sqrt{\varepsilon})\,.$$

Again, $|\beta| \to \infty$ is impossible and (40) holds. In order to estimate the strength of S_2, we can use the same arguments to prove

$$\sup g_2'(|\chi|, \rho_0) =: C_4 < 1, \tag{41}$$

and

$$\frac{1 - g_2'(|\chi|, \rho_0)}{g_2'(|\chi|, \rho_0)} \geq \frac{1 - C_4}{C_4} =: C_5 > 0\,. \tag{42}$$

From now on, we shall put

$$C_0 = \min\{C_3, C_5\}\,. \tag{43}$$

4. Global interaction estimates

This section contains all the necessary assertions from [1] with several changes in constants. All changes are similar to those from the previous section.

Definition 4.1. [1] *A Lipschitz curve J defined by $t = T(x)$, $x \in \mathbb{R}$ is called an I-curve, if $|T'(x)| < 1/\hat{\lambda}$. We denote $J_2 > J_1$, if $T_1 \neq T_2$ and $T_2(x) \geq T_1(x), x \in \mathbb{R}$. Denoting by $S_j(J)$ the set of j-shock waves crossing J and $S(J) = S_1(J) + S_2(J)$, we define*

$$L^-(J) = \sum_{\alpha \in S(J)} |\alpha|, \quad Q(J) = \sum_{\beta \in S_1(J), \chi \in S_2(J),\ \beta,\chi \text{ approach}} |\beta||\chi|\,. \tag{44}$$

Set $F(J) = L^-(J) + \tilde{K} \cdot Q(J)$, where $\tilde{K} := 4C_*\sqrt{\varepsilon}$. A space-like line lying between the initial line and the first interaction point is denoted with O.

Lemma 4.2.

$$Q(O) \leq L^-(O)^2\,. \tag{45}$$

Proof. The proof follows straightforward from Definition 4.1. □

Lemma 4.3. *Assuming $4\,C_*\,\sqrt{\varepsilon}\,L^-(O) \leq 1$, we have*

$$F(O) \leq 2L^-(O)\,. \tag{46}$$

Proof.

$$\begin{aligned} F(O) = &\ L^-(O) + \tilde K\, Q(O) \le L^-(O) + \tilde K\, L^-(O)^2 \text{ (by (45))} \\ = &\ L^-(O)(1+\tilde K L^-(O)) = L^-(O)(1+4\, C_* \sqrt{\varepsilon}\, L^-(O)) \\ \le &\ L^-(O)(1+1) = 2L^-(O)\ . \end{aligned}$$

□

As in [2], consider a interval $\mathcal{J} \subset \mathbb{R}$ and a map $a : \mathcal{J} \to \mathbb{R}^n$. The *total variation* (TV) of a is then defined as

$$TV(a) := \sup \left\{ \sum_{j=1}^{N} |a(x_j) - a(x_{j-1})| \right\},$$

where the supremum is taken over all $N \ge 1$ and all $(N+1)$-tuples of points $x_j \in \mathcal{J}$ such that $x_0 < x_1 < \cdots < x_N$. Now, we give a new estimate for $L^+(O)$. Here, $L^+(O)$ denotes the sum of the rarefaction waves strengths which cross the line O.

Lemma 4.4. *We have*

$$L^-(O) \le TV(r_0(x), s_0(x)) \quad \text{and} \quad L^+(O) \le TV(r_0(x), s_0(x))\ . \tag{47}$$

The estimates in previous Lemma can easily be verified. The uniform bounds of $F(J)$ follows from the following theorem.

Theorem 4.5. *If* $C_* \sqrt{\varepsilon}\, F(O) \le \min\left\{\frac{1}{2}, \frac{C_0}{4}\right\}$, *then* $F(J_2) \le F(J_1)$ *for* $J_2 > J_1$. *Particulary,* $L^-(J) \le F(O)$.

Proof. This theorem can be proved in the same way as Lemma 5 from [11] and hence the proof will be omitted. One has just to substitute a constant K from the original proof with the determined value $\tilde K$ here. □

Lemma 4.6. *Assume that* $\tilde K L^-(O) \le 1$ *and that*

$$\sqrt{\gamma - 1}\, TV\,(r_0(x), s_0(x)) \le \frac{1}{C_*} \cdot \min\left\{ \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{8} C_0 \right\}\ . \tag{48}$$

Then $\tilde K F(O) \le \min\{2, C_0\}$.

Proof. Using Lemma 4.3 and 4.4 we have

$$\frac{\sqrt{2}}{2}\sqrt{\varepsilon}\, F(O) \le \sqrt{2}\sqrt{\varepsilon}\, L^-(O) \le \sqrt{\gamma-1}\, TV\,(r_0(x), s_0(x)) \le \frac{1}{C_*} \cdot \min\left\{ \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{8} C_0 \right\}\ .$$

Multiplying it with $8\, C_*/\sqrt{2}$, one gets

$$4\, C_* \sqrt{\varepsilon}\, F(O) = \tilde K F(O) \le \min\{2, C_0\}\ .$$

which proves the claim. □

The right hand side of (48) does not depend on ε, and then one can say that $TV\,(r_0(x), s_0(x))$ may be arbitrarily large since we can always choose ε small enough in order to fulfill (48) with $\gamma = 1 + 2\varepsilon$. Then we can apply wave front tracking procedure from [1] for each such ε, and obtain a sequence of step functions converging to the entropic solution. One only needs to replace $C\varepsilon F(O)$ and $\frac{1-\delta}{\delta}$ from [1] with $C_* \sqrt{\varepsilon} F(O)$ and C_0, respectively.

5. Approximate delta shock solutions to pressureless gas dynamics

Our main task is to solve delta shock interaction problem for pressureless gas dynamics model. Accordingly, we will introduce a solution concept from [9] (somewhat simplified) and check consistency of theoretical and numerical wave front tracking results by letting $\varepsilon \to 0$.

5.1. **Basic notions.**

In this section we shall use the notions and assertions from [9]. It contains results for a 3×3 system with energy conservation law added, but all the results can also be applied to system (2), too. Let us start with the basic definitions. Vector valued function of the form

$$U_\varepsilon(x,t) = \begin{cases} U_0, & x < c(t) - a_\varepsilon(t) \\ U_{1,\varepsilon}(t), & c(t) - a_\varepsilon(t) < x < c(t) \\ U_{2,\varepsilon}(t), & c(t) < x < c(t) + b_\varepsilon(t) \\ U_1, & x > c(t) + b_\varepsilon(t) \end{cases}. \tag{49}$$

is called *weighted shadow wave* (weighted SDW, for short). Here, $U := (\rho, u)$. The functions a_ε, b_ε are continuous functions satisfying $a_\varepsilon(0) = x_{1,\varepsilon}$ and $b_\varepsilon(0) = x_{2,\varepsilon}$. The SDW is *constant* if $U_{1,\varepsilon}$ and $U_{2,\varepsilon}$ are just constants. If, in addition, $x_{1,\varepsilon} = x_{2,\varepsilon} = 0$, then the wave is called *simple*.
The value

$$\sigma_\varepsilon(t) := a_\varepsilon(t) U_{1,\varepsilon}(t) + b_\varepsilon(t) U_{2,\varepsilon}(t)$$

is called the *strength* and $c'(t)$ is called the *speed* of the shadow wave. We assume that $\lim_{\varepsilon\to 0} \sigma_\varepsilon(t) = \sigma(t) \in \mathbb{R}^n$ exists for every $t \geq 0$ and

$$\lim_{\varepsilon\to 0} \int U_\varepsilon(x,t)\phi(x,t)\,dx\,dt = \langle U_0 + (U_1 - U_0)\,\theta(x - c(t)) + \sigma(t)\,\delta(x - c(t)), \phi(x,t)\rangle,$$

for $t \geq 0$, where θ is a Heaviside function. The SDW *central line* is given by $x = c(t)$, while $x = c(t) - a_\varepsilon(t)$ and $x = c(t) + b_\varepsilon(t)$ are called the *external SDW lines*. The values $x_{1,\varepsilon}$ and $x_{2,\varepsilon}$ are called the *shifts*, while $U_{1,\varepsilon}(t)$ and $U_{2,\varepsilon}(t)$ are called the *intermediate states* of a given SDW.
Let $i \in \{1, 2, \ldots, n\}$. We assume $\|U^i_\varepsilon\|_{L^\infty} = \mathcal{O}(\varepsilon^{-1})$, if f and g have at most a linear growth with respect to i-th component, or otherwise $\|U^i_\varepsilon\|_{L^\infty} = o(\varepsilon^{-1})$. The components of the first kind are called *major* ones, while the ones of the second kind are called *minor* ones.
A *delta shock* is a SDW associated with a δ distribution with all minor components having finite limits as $\varepsilon \to 0$.

The following lemma is the base of all calculations involving SDWs.

Lemma 5.1. *Let $f, g \in \mathcal{C}(\Omega : \mathbb{R}^n)$ and $U : \mathbb{R}^2_+ \to \Omega \subset \mathbb{R}^n$ be a piecewise constant function for every $t \geq 0$. Let us also suppose that f and g satisfy*

$$\max_{i=1,2}\{\|f(U_{i,\varepsilon})\|_{L^\infty}, \|g(U_{i,\varepsilon})\|_{L^\infty}\} = \mathcal{O}(\varepsilon^{-1}). \tag{50}$$

Then

$$\begin{aligned}\langle \partial_t f(U_\varepsilon), \phi\rangle \approx &\int_0^\infty \lim_{\varepsilon\to 0} \frac{d}{dt}\Big(a_\varepsilon(t) f(U_{1,\varepsilon}(t)) + b_\varepsilon(t) f(U_{2,\varepsilon}(t))\Big)\,\phi(c(t),t)\,dt \\ &- \int_0^\infty c'(t)\Big(f(U_1) - f(U_0)\Big)\,\phi(c(t),t)\,dt \\ &+ \int_0^\infty \lim_{\varepsilon\to 0} c'(t)\Big(a_\varepsilon(t) f(U_{1,\varepsilon}(t)) + b_\varepsilon(t) f(U_{2,\varepsilon}(t))\Big)\,\partial_x\phi(c(t),t)\,dt\end{aligned} \tag{51}$$

and

$$\begin{aligned}\langle \partial_x g(U_\varepsilon), \phi\rangle \approx &\int_0^\infty \Big(g(U_1) - g(U_0)\Big)\,\phi(c(t),t)\,dt \\ &- \int_0^\infty \lim_{\varepsilon\to 0} \Big((a_\varepsilon(t) g(U_{1,\varepsilon}(t)) + (b_\varepsilon(t) g(U_{2,\varepsilon}(t))\Big)\,\partial_x\phi(c(t),t)\,dt.\end{aligned} \tag{52}$$

5.2. **Entropy conditions.** Let $\eta(U)$ be a *semi-convex* entropy function for (2), with entropy-flux function $q(U)$. We shall use entropy condition in the following form. A weak or approximate solution $U_\varepsilon = (\rho_\varepsilon, u_\varepsilon)$ to system (2) with initial data $U|_{t=0} = U_{0,\varepsilon}$ is *admissible* provided that for every $T > 0$ we have

$$\underline{\lim}_{\varepsilon\to 0} \int_{\mathbb{R}}\int_0^T \eta(U_\varepsilon)\partial_t\phi + q(U_\varepsilon)\partial_x\phi\,dt\,dx + \int_{\mathbb{R}} \eta(U_{0,\varepsilon}(x,0))\phi(x,0)\,dx \geq 0, \tag{53}$$

for all non-negative test functions $\phi \in C_0^\infty(\mathbb{R}\times(-\infty,T))$.

Using Lemma 5.1 with f substituted by η and g by q and the fact that the delta function is a non-negative distribution, the first condition for SDW U_ε from (49) to be admissible is given by

$$\begin{aligned}&-c'(t)(\eta(U_1) - \eta(U_0)) + (q(U_1) - q(U_0)) \\ &\quad + \lim_{\varepsilon\to 0} \frac{d}{dt}(\eta(U_{1,\varepsilon}(t))a_\varepsilon + \eta(U_{2,\varepsilon}(t))b_\varepsilon) \leq 0.\end{aligned} \tag{54}$$

The derivative of delta function changes the sign, so U_ε has to satisfy

$$\begin{aligned}&\lim_{\varepsilon\to 0} c'(t)(\eta(U_{1,\varepsilon}(t))a_\varepsilon + \eta(U_{2,\varepsilon}(t))b_\varepsilon) \\ &\quad - q(U_{1,\varepsilon}(t))a_\varepsilon(t) - q(U_{2,\varepsilon}(t))b_\varepsilon(t) = 0\end{aligned} \tag{55}$$

in addition.

These conditions are much simpler in the case of simple SDW when U_0, U_1, $U_{1,\varepsilon}$ and $U_{2,\varepsilon}$ are constants:

$$\overline{\lim}_{\varepsilon\to 0} - c(\eta(U_1) - \eta(U_0)) + a_\varepsilon\eta(U_{1,\varepsilon}) + b_\varepsilon\eta(U_{2,\varepsilon}) + q(U_1) - q(U_0) \leq 0 \tag{56}$$

and

$$\lim_{\varepsilon\to 0} -c(a_\varepsilon\eta(U_{1,\varepsilon}) + b_\varepsilon\eta(U_{2,\varepsilon})) + a_\varepsilon q(U_{1,\varepsilon}) + b_\varepsilon q(U_{2,\varepsilon}) = 0. \tag{57}$$

In most of the papers with delta or singular shock solution, the authors use over-compressibility as the admissibility condition. A wave is called the *overcompressive* one if all characteristics from both sides of the SDW line run into a shock curve, i.e.

$$\lambda_i(U_0) \geq c'(t) \geq \lambda_i(U_1),\ i = 1,\dots,n,$$

where c is a shock speed and $x = \lambda_i(U)t$, $i = 1,\dots,n$ are the characteristics of the system. One will see that these notations coincide with our model case.

The entropy condition is connected with the problem of uniqueness for a weak solution of the conservation law system. We give a definition of weak (distributional) uniqueness and some results about it afterward.

Definition 5.2. *An SDW solution is called weakly unique if its distributional image is unique. More precisely, a speed c of the wave has to be unique as well as the limit*

$$\lim_{\varepsilon \to 0} a_\varepsilon U_{1,\varepsilon} + b_\varepsilon U_{2,\varepsilon}.$$

Let $i \in \{1, \ldots, n\}$. If a limit $\lim_{\varepsilon \to 0} a_\varepsilon U^i_{1,\varepsilon} + b_\varepsilon U^i_{2,\varepsilon}$ is unique, then we say that the i-th component is unique.

Note that all minor components of U_ε are unique by default.

5.3. **Entropy solutions to Riemann problem for pressureless gas dynamics model.** The proof for the following theorem in the case of 3×3 PGD model is given in [9]. Its restriction to a 2×2 system is straightforward and therefore not discussed here.

Theorem 5.3. *Suppose that $u_0 > u_1$. Then there exists a unique shadow wave solution of the form (49) to the Riemann problem (2, 3) satisfying the entropy inequality (53) with η and q as defined above.*
Moreover, the validity of (53) for all semi-convex entropies η are equivalent to the overcompressibility of the shadow wave.

Our aim is to show the structure of a solution in order to be able to compare it with a numerical approximation described above. For our purposes it is safe to take $a_\varepsilon = b_\varepsilon = \varepsilon$ in the sequel. In the proof of Theorem 5.3 we showed that a SDW solution (49) (with $U = (\rho, u)$) to (2) and initial data (3), with $u_0 > u_1$, had to satisfy

$$c = u_s = \lim_{\varepsilon \to 0} u_\varepsilon \equiv \frac{[\rho u] - [u]\sqrt{\rho_0 \rho_1}}{[\rho]} \quad (u_s \text{ does not depend on } \varepsilon)$$
$$\lim_{\varepsilon \to 0} \varepsilon \rho_\varepsilon = c[\rho] - [\rho u] = (u_0 - u_1)\sqrt{\rho_0 \rho_1},$$

if $\rho_0 \neq \rho_1$, and $c = u_s = (u_0 + u_1)/2$, if $\rho_0 = \rho_1$. That defines a weakly unique SDW solution to the problem.

5.4. **Two SDWs interaction.** The main advantage of using weighted SDWs (intermediate states vary with t in addition) is for solving SDW interaction problem. Then we can proceed with the main part of the paper by showing numerically that such a solution can be viewed as a limit of gas dynamics model with a vanishing pressure as perturbation. Note that verification of delta shock existence has already been obtained in [3] (see [6] for a somewhat general model).
Suppose that two SDWs interact in a point (X, T). The superscript 1 is used for data in the left wave while the superscript 2 is used for the right one. The first SDW connects the states $U_0 = (\rho_0, u_0)$ with $U_1 = (\rho_1, u_1)$, while the second one connects the states $U_1 = (\rho_1, u_1)$ with $U_2 = (\rho_2, u_2)$.
Again, the following theorem has been proved in [9] for the extended PGD system, and the proof can easily be adopted for the present one (2).

Theorem 5.4. *The result of two SDW interactions for the pressureless system (2) is a weakly unique single entropic weighted SDW.*

We use the following notation: $[x]_1 := x_1 - x_0$, $[x]_2 := x_2 - x_1$ and $[x] := x_2 - x_0$. The weighted SDW solution from the above theorem satisfies the following: The speed is given by $c'(t) = u_s(t) := \lim_{\varepsilon\to 0} u_\varepsilon(t)$, while $u_s(t)$ and $\xi(t) := \lim_{\varepsilon\to 0} \varepsilon\rho_\varepsilon(t)$ satisfies the following ODEs system

$$\begin{aligned} \xi'(t) &= u_s(t)[\rho] - [\rho u] \\ (\xi(t)u_s(t))' &= u_s(t)[\rho u] - [\rho u^2] \end{aligned} \tag{58}$$

with the initial data

$$\begin{aligned} \xi(T) =&(\xi^1 + \xi^2)T = (-[u]_1\sqrt{\rho_0\rho_1} - [u]_2\sqrt{\rho_1\rho_2})T, \\ \xi(T)u_s(T) =&(c^1\xi^1 + c^2\xi^2)T = \Big(- \frac{[\rho u]_1 - [u]_1\sqrt{\rho_0\rho_1}}{[\rho]_1} \cdot [u]_1\sqrt{\rho_0\rho_1} \\ &- \frac{[\rho u]_2 - [u]_2\sqrt{\rho_1\rho_2}}{[\rho]_2} \cdot [u]_2\sqrt{\rho_1\rho_2}\Big)T. \end{aligned} \tag{59}$$

Here are some facts regarding the solution $(\xi(t), u_s(t))$, $t \geq T$ to the above initial data problem (see [9]):

(1) $\xi(t)$, for $t > T$, is an increasing function when exists. The initial data $\xi(T) > 0$ and $\xi(t)$ is always positive function for $t > T$ (when exists), since $u_0 > u_1 > u_2$.

(2) From the system (58) we have

$$u_s'(t) = -\frac{1}{\xi(t)}([\rho]u_s^2(t) - 2[\rho u]u_s(t) + [\rho u^2]).$$

The value $-1/\xi(t)$ is now always negative for $t > T$. The roots of the right-hand side of the above ODE are denoted as $A_1 < A_2$. Then, for $[\rho] \neq 0$,

$$A_{1,2} = \frac{[\rho u] \pm |u_0 - u_2|\sqrt{\rho_0\rho_2}}{[\rho]}.$$

Assume that $[\rho] > 0$. If $u_s(t) \in (A_1, A_2)$, then $u_s(t)$ increases, and if $u_s(t) \in (-\infty, A_1) \cup (A_2, +\infty)$, then $u_s(t)$ decreases. The opposite holds if $[\rho] < 0$. There are two possible cases:

- If $\rho_0 > \rho_2$, then $u_2 \leq A_1 \leq u_0 \leq A_2$. If $u_s(T) \in (u_2, A_1)$, then $u_s(t)$ increases for $t > T$ but stays bellow A_1. If $u_s(T) \in (A_1, u_0)$, then $u_s(t)$ decreases for $t > T$ but stays above A_1.
- If $\rho_2 > \rho_0$, then $A_1 \leq u_2 \leq A_2 \leq u_0$. Again, if $u_s(T) \in (u_2, A_2)$, then $u_s(t)$ increases for $t > T$ but stays bellow A_2. If $u_s(T) \in (A_2, u_0)$, then $u_s(t)$ decreases for $t > T$ but stays above A_2.

This implies $u_0 \geq u_s(t) \geq u_2$ (the SDW is overcompressive). Also, one will see that numerical examples resemble these asymptotic properties of $u_s(t)$ as $t \to \infty$.

6. Numerical results

In this section one can find numerical results which show a consistency of theoretical (in the sense of SDWs) and numerical results. Consider system (1) with the initial data

$$(\rho, u)\,|_{t=0} = \begin{cases} (\rho_0, u_0), & x < a_1 \\ (\rho_1, u_1), & a_1 < x < a_2 \\ (\rho_2, u_2), & x > a_2 \end{cases} \tag{60}$$

TABLE 1. Parameter description

Parameter	Description
κ	Adiabatic constant defined in (4).
ρ_ε	First component of the intermediate state of the solution for (1, 60).
u_ε	Second component of the intermediate state of the solution for (1, 60).
c_1	Speed of the first left shock.
c_2	Speed of the last right shock.
$\lvert Eq_1\rvert$	Left hand side of the integral on the first equation in (1).
$\lvert Eq_2\rvert$	Left hand side of the integral on the second equation in (1).

where $a_1 < a_2$, $u_0 > u_1 > u_2$. Then (see [3]), for ε small enough, there exist $(\rho_{1,\varepsilon}, u_{1,\varepsilon}) \in \mathbb{R}_+ \times \mathbb{R}$ and $(\rho_{2,\varepsilon}, u_{2,\varepsilon}) \in \mathbb{R}_+ \times \mathbb{R}$, so that:

- (ρ_0, u_0) is connected with $(\rho_{1,\varepsilon}, u_{1,\varepsilon})$ by an 1-shock, and $(\rho_{1,\varepsilon}, u_{1,\varepsilon})$ is connected with (ρ_1, u_1) by a 2-shock,
- (ρ_1, u_1) is connected with $(\rho_{2,\varepsilon}, u_{2,\varepsilon})$ by an 1-shock, while $(\rho_{2,\varepsilon}, u_{2,\varepsilon})$ is connected with (ρ_2, u_2) by a 2-shock.

A numerical solution is obtained by wave front tracking algorithm described in [1]. In order to verify two delta shocks interaction, we shall consider two cases.

Case A. Suppose that (ρ_0, u_0) is connected with (ρ_1, u_1) by a single delta shock and (ρ_1, u_1) is connected with (ρ_2, u_2) by a single delta shock, too. Assume that (ρ_0, u_0) can be connected with (ρ_2, u_2) by a single delta shock (so-called simple SDW, see [9]). The resulting SDW has a constant speed as a consequence. That can be done by choosing a special value for ρ_2 provided that ρ_0, u_0, ρ_1, u_1 and u_2 are already given.
Case B. We choose arbitrarily ρ_2, i.e. the resulting SDW has a variable speed (a central SDW curve is no longer a line). The numerical results are given in Tables 2, 3 and 4.

6.1. Case A.

Example 6.1. Let $a_1 = 0$, $a_2 = 2$, $(\rho_0, u_0) = (1, 1)$, $(\rho_1, u_1) = (1.2, 0.8)$ and $u_2 = 0.7$. Now, for $\rho_2 = 1.14286$, there exists a single simple SDW as a solution to the interaction problem.

TABLE 2. $(\rho_0, u_0) = (1, 1), (\rho_1, u_1) = (1.2, 0.8), (\rho_2, u_2) = (1.14286, 0.7)$

γ	κ	ε	ρ_ε	u_ε	c_1	c_2	$\lvert Eq_1\rvert$	$\lvert Eq_2\rvert$
2	0.5	0.5	1.29979	0.80062	0.13553	1.53337	$2 \cdot 10^{-5}$	$3 \cdot 10^{-5}$
1.2	0.29	0.1	1.68612	0.82806	0.57746	1.09746	$1 \cdot 10^{-4}$	$2 \cdot 10^{-4}$
1.02	0.099	0.01	4.22718	0.84163	0.79256	0.89411	$2 \cdot 10^{-3}$	$3 \cdot 10^{-3}$
1.01	0.071	0.005	6.71337	0.84311	0.81565	0.87247	$8 \cdot 10^{-5}$	$2 \cdot 10^{-3}$
1.006	0.055	0.003	9.95729	0.84379	0.82636	0.86254	$1 \cdot 10^{-2}$	$6 \cdot 10^{-3}$

After interaction, the speed of the resulting wave is $c_\delta = 0.844994$. Two SDWs will interact in a point $(X, T) = (12.365, 13.8088)$ with such data. Now, we are going to

explain Figures 2, 7 and 12 which are illustrations of appropriate numerical results. For each ε we have two piecewise linear half-lines. The left one originates from the point $(x,t) = (a_1, 0)$, while the right one originates from the point $(x,t) = (a_2, 0)$. The i-th linear segment of these half-lines can be written in the form $x = c_{i,j}(t - t_i) + x_i$, $i \geq 1$, $j = 1, 2$, $x_i \leq x \leq x_{i+1}$, $t_i \leq t \leq t_{i+1}$, where $c_{i,1}$ stands for the speed of the first (S_1) wave on the left hand side in phase plane, while $c_{i,2}$ stands for the speed of the last (S_2) wave on the left hand side in phase plane at each i-th segment. Interactions of the waves occur at the points (x_i, t_i), $i \geq 1$. After two delta shock interaction, the resulting delta shock central line in Figure 2 (dashed line) starts from (X, T) and it is calculated explicitly from system (2).

6.2. **Case B.**

Example 6.2. Let $a_1 = 0$, $a_2 = 2$, $(\rho_0, u_0) = (1, 1)$, $(\rho_1, u_1) = (1.2, 0.8)$ and $(\rho_2, u_2) = (1.3, 0.7)$. Two SDWs will interact in a point $(X, T) = (12.2291, 13.657)$ with such data. After two delta shock interaction, the resulting delta shock central lines in Figures 7 and 12 (dashed lines) start from (X, T), too. Here,

$$x(t) = \int_T^t u_s(p)\, dp + X, \ t \geq T,$$

while $u_s(t)$ represents the second component of the solution $(\xi(t), u_s(t))$ of system (58) with initial conditions (59).

Table 3. $(\rho_0, u_0) = (1, 1), (\rho_1, u_1) = (1.2, 0.8), (\rho_2, u_2) = (1.3, 0.7)$

γ	κ	ε	ρ_ε	u_ε	c_1	c_2	$\lvert Eq_1\rvert$	$\lvert Eq_2\rvert$
2	0.5	0.5	1.38131	0.74968	0.09312	1.54395	$3 \cdot 10^{-4}$	$5 \cdot 10^{-4}$
1.2	0.29	0.1	1.79287	0.80661	0.56268	1.08778	$5 \cdot 10^{-4}$	$6 \cdot 10^{-4}$
1.02	0.099	0.01	4.49667	0.83356	0.78596	0.88787	$3 \cdot 10^{-3}$	$3 \cdot 10^{-3}$
1.01	0.071	0.005	7.14038	0.83646	0.80983	0.86684	$1 \cdot 10^{-3}$	$3 \cdot 10^{-3}$
1.006	0.055	0.003	10.5884	0.83782	0.82091	0.85711	$3 \cdot 10^{-2}$	$2 \cdot 10^{-2}$

Example 6.3. Let $a_1 = 0$, $a_2 = 2$, $(\rho_0, u_0) = (1, 1)$, $(\rho_1, u_1) = (0.8, 0.9)$ and $(\rho_2, u_2) = (0.9, 0.7)$. Two SDWs will interact in a point $(X, T) = (12.2364, 12.8427)$ with such data.

Table 4. $(\rho_0, u_0) = (1, 1), (\rho_1, u_1) = (0.8, 0.9), (\rho_2, u_2) = (0.9, 0.7)$

γ	κ	ε	ρ_ε	u_ε	c_1	c_2	$\lvert Eq_1\rvert$	$\lvert Eq_2\rvert$
2	0.5	0.5	1.16621	0.88674	0.20529	1.51813	$8 \cdot 10^{-5}$	$4 \cdot 10^{-5}$
1.2	0.29	0.1	1.50419	0.86711	0.60356	1.11606	$2 \cdot 10^{-4}$	$1 \cdot 10^{-4}$
1.02	0.099	0.01	3.75748	0.85659	0.80459	0.90592	$4 \cdot 10^{-3}$	$2 \cdot 10^{-3}$
1.01	0.071	0.005	5.96447	0.85544	0.82632	0.88306	$1 \cdot 10^{-3}$	$8 \cdot 10^{-4}$
1.006	0.055	0.003	8.66370	0.85341	0.83808	0.86341	$2 \cdot 10^{-2}$	$3 \cdot 10^{-2}$

References

[1] F. Asakura, *Wave-front tracking method for the equations of isentropic gas dynamics*, Quart. Appl. Math. 63 (2005), no. 1, 20–33.

[2] A. Bressan, *Hyperbolic Systems of Conservation Laws. The One-Dimensi-onal Cauchy Problem*, Oxford University Press, New York, 2000.

[3] G.Q. Chen and H. Liu, *Formation of delta-shocks and vacuum states in the vanishing pressure limit of solutions to the isentropic Euler equations*, SIAM J. Math. Anal. 34 (2003), no. 4, 925–938.

[4] R. Courant and K. Friedrichs, *Supersonic Flow and Shock Waves*, Applied Mathematical Sciences, Vol. 21. Springer-Verlag, New York-Heidelberg, 1976.

[5] R.J. LeVeque, *Numerical Methods for Conservation Laws*, Birkhäuser Verlag, Basel, 1990.

[6] D. Mitrović and M. Nedeljkov, *Delta shock waves as a limit of shock waves*, J. Hyperbolic Differ. Equ. 4 (2007), no. 4, 629–653.

[7] M. Nedeljkov, *Delta and singular delta locus for one dimensional systems of conservation laws*, Math. Methods Appl. Sci. 27 (2004), no. 8, 931–955.

[8] M. Nedeljkov, *Singular shock waves in interactions*, Quart. Appl. Math. 66 (2008), no. 2, 281–302.

[9] M. Nedeljkov, Shadow waves – entropies and interactions for delta and singular shocks (2009), to appear in Arch. Ration. Mech. Anal.

[10] M. Nedeljkov, M. Oberguggengberger, *Interactions of delta shock waves in a strictly hyperbolic system of conservation laws*, J. Math. Anal. Appl. 344 (2008), no. 2, 1143–1157.

[11] T. Nishida, J.A. Smoller, *Solutions in the Large for Some Nonlinear Hyperbolic Conservation Laws*, Comm. Pure Appl. Math. 26 (1973), 183–200.

[12] B. Riemann, *Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite*, Gott.Abh.Math.Cl. 8 (1860), 43–65.

[13] E. Weinan, Y.G. Rykov, Ya.G. Sinai, *Generalized variotional principles, global weak solutions and behavior with random initial data for systems of consevation laws arising in adhesion particle dynamics*, Comm. Math. Phys. 177 (1996), no. 2, 349–380.

[14] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York, 1994.

Nebojša Dedović
Department of Agricultural Engineering, University of Novi Sad
Trg D. Obradovića 8, 21000 Novi Sad, Serbia
dedovicn@uns.ac.rs

Marko Nedeljkov
Department of Mathematics and Informatics, University of Novi Sad
Trg D. Obradovića 4, 21000 Novi Sad, Serbia
markonne@uns.ac.rs

7. Appendix

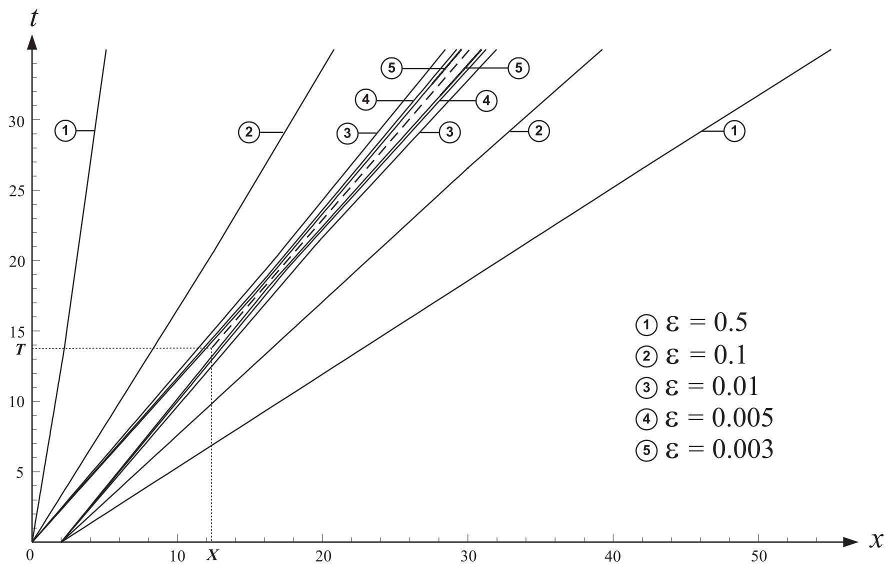

Figure 2. Phase $x-t$ plane, Case A, Example 6.1.

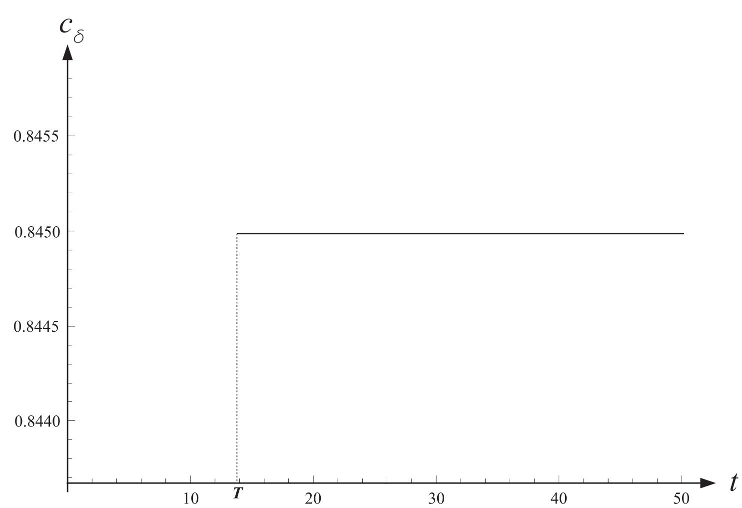

Figure 3. Speed of delta shock formed after double delta shock interaction, Case A, Example 6.1.

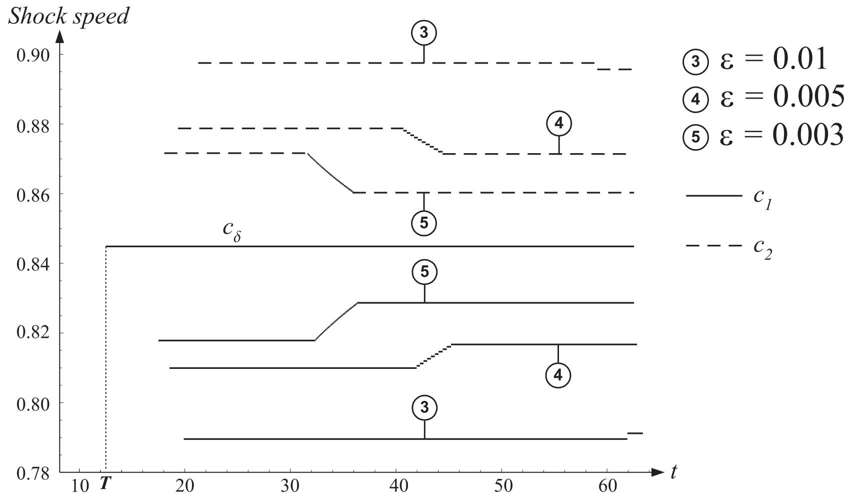

FIGURE 4. Speed of the first (S_1) wave on the left hand side and the last (S_2) wave on the right hand side for various ε, Case A, Example 6.1.

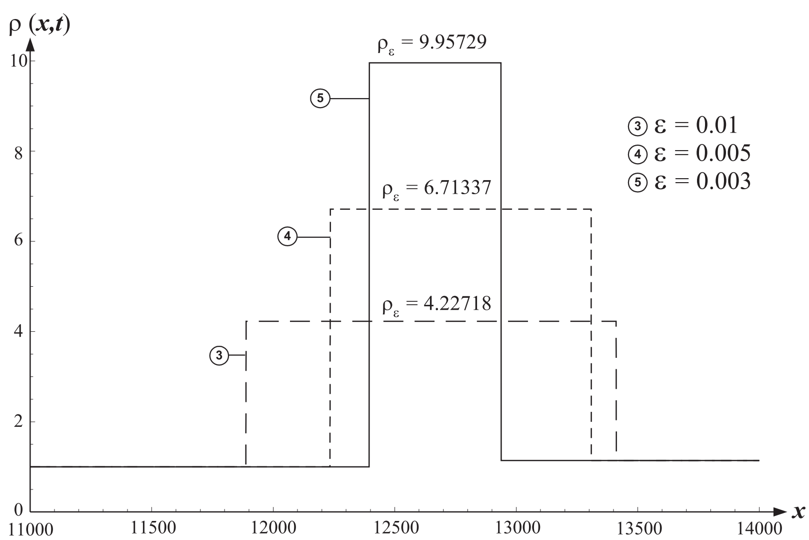

FIGURE 5. Solution $\rho(x,t)$ for various ε at $t = 15000$, Case A, Example 6.1.

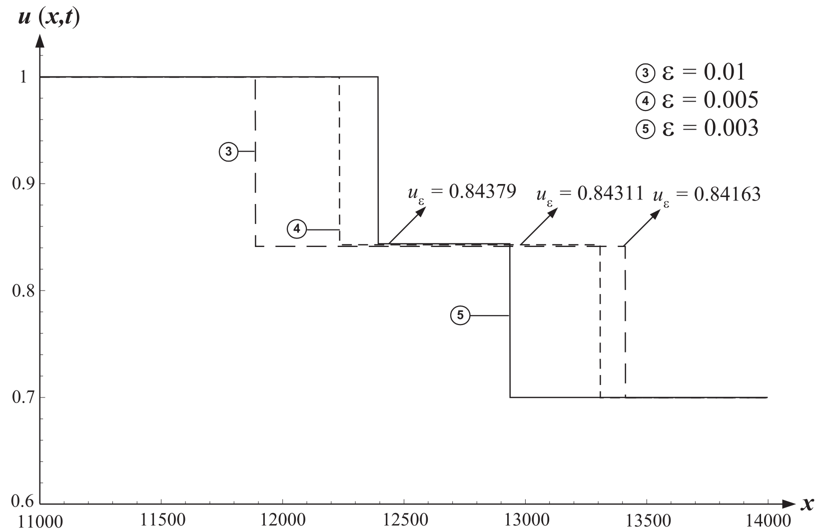

FIGURE 6. Solution $u(x,t)$ for various ε at $t = 15000$, Case A, Example 6.1.

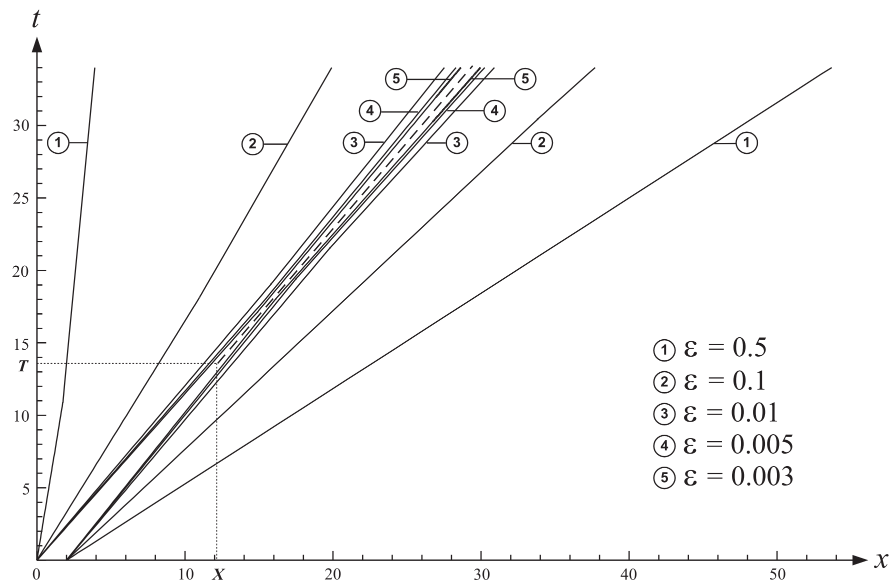

FIGURE 7. Phase $x - t$ plane, Case B, Example 6.2.

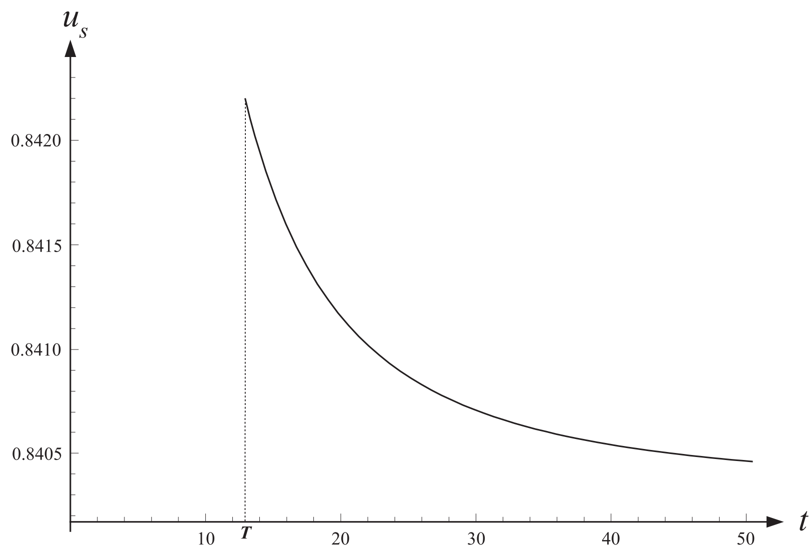

FIGURE 8. Speed of delta shock formed after double delta shock interaction, Case B, Example 6.2.

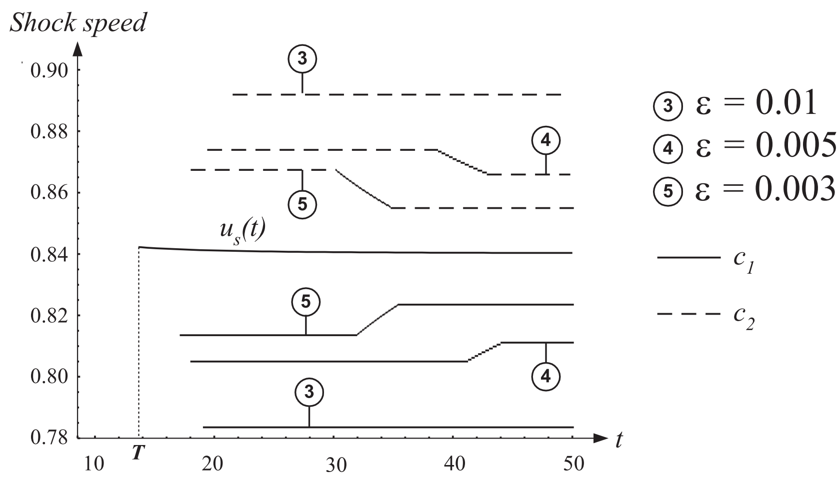

FIGURE 9. Speed of the first (S_1) wave on the left hand side and the last (S_2) wave on the right hand side for various ε, Case B, Example 6.2.

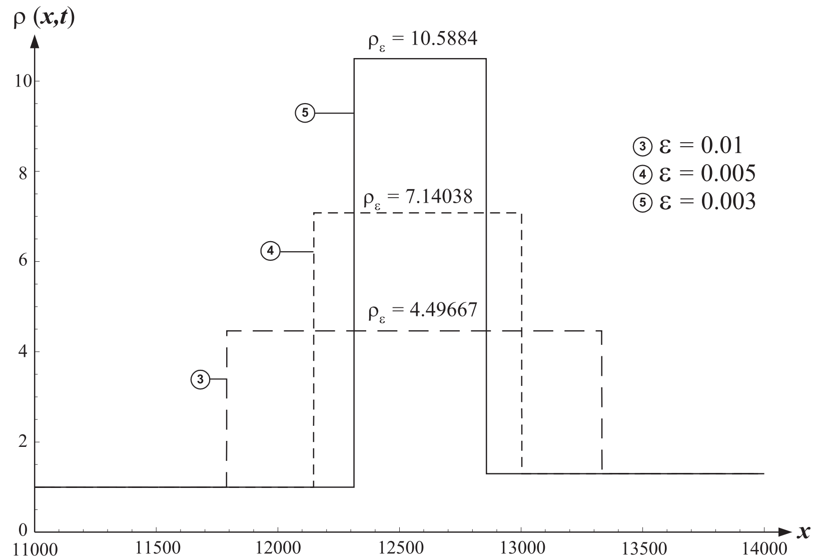

FIGURE 10. Solution $\rho(x,t)$ for various ε at $t = 15000$, Case B, Example 6.2.

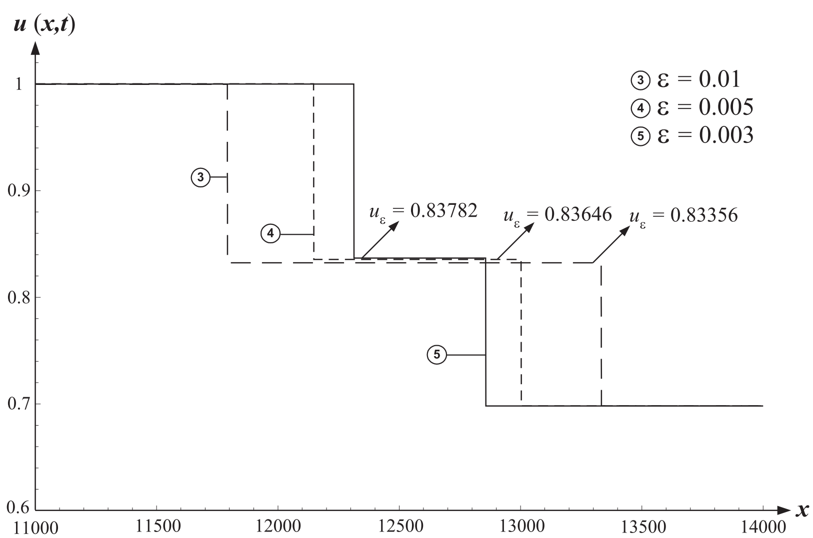

FIGURE 11. Solution $u(x,t)$ for various ε at $t = 15000$, Case B, Example 6.2.

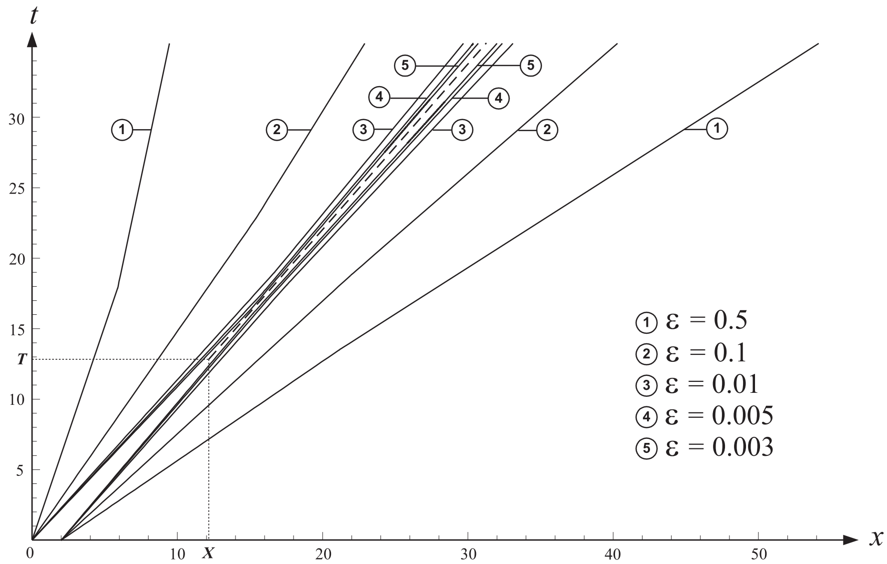

FIGURE 12. Phase $x - t$ plane, Case B, Example 6.3.

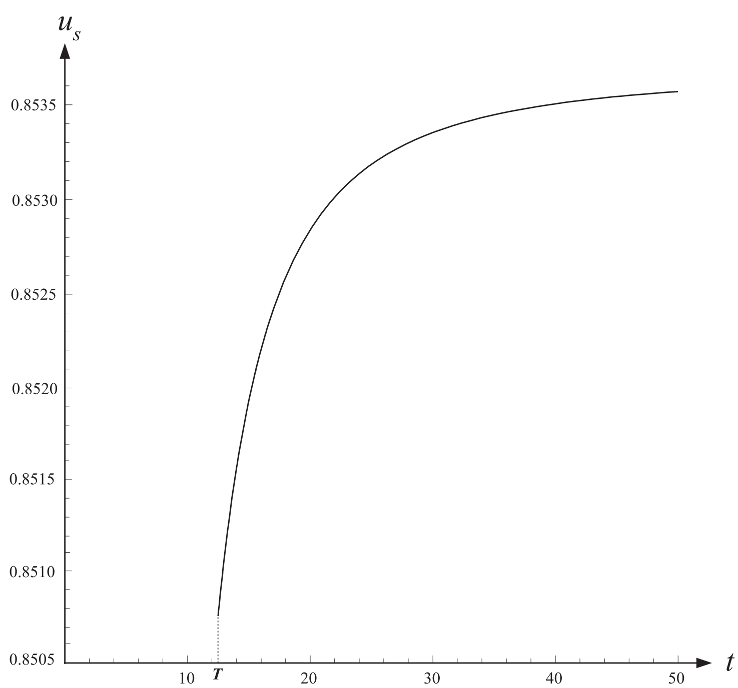

FIGURE 13. Speed of delta shock formed after double delta shock interaction, Case B, Example 6.3.

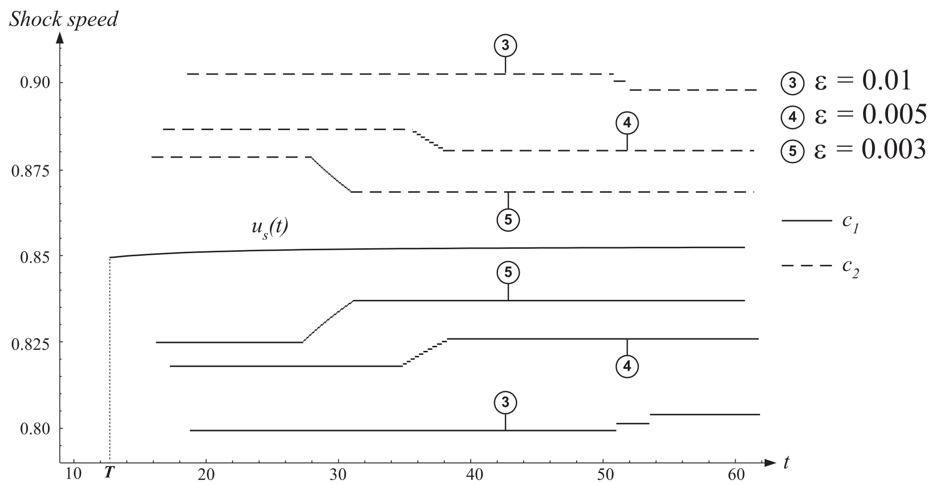

FIGURE 14. Speed of the first (S_1) wave on the left hand side and the last (S_2) wave on the right hand side for various ε, Case B, Example 6.3.

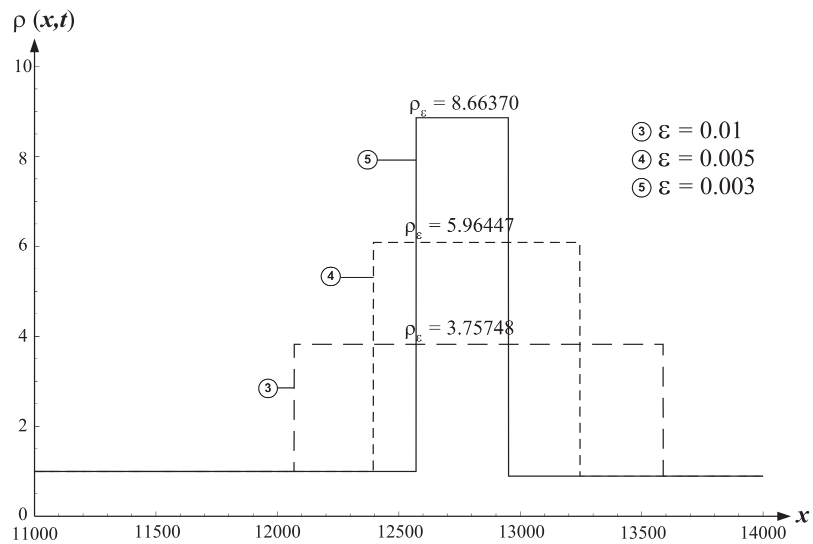

FIGURE 15. Solution $\rho(x,t)$ for various ε at $t = 15000$, Case B, Example 6.3.

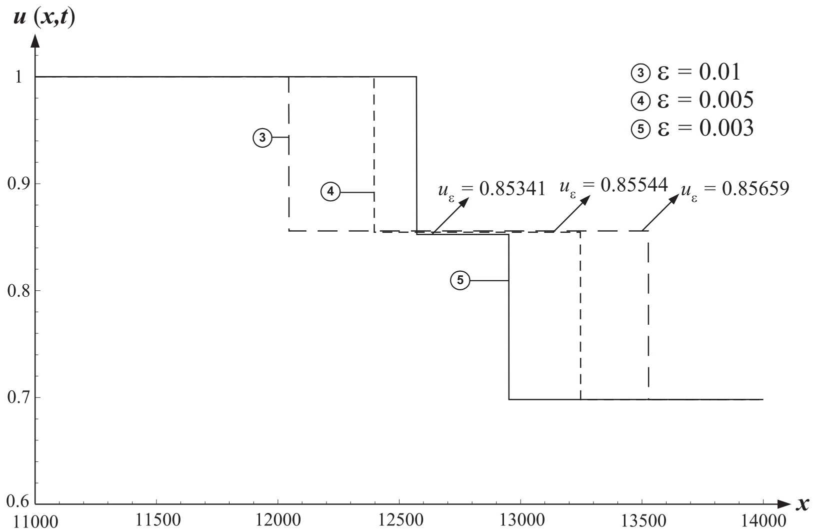

FIGURE 16. Solution $u(x,t)$ for various ε at $t = 15000$, Case B, Example 6.3.